

Correspondences and their Action on Filtrations on Cohomology and 0-cycles of Abelian Varieties

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Abstract

We prove that, given a symmetrically distinguished correspondence of a suitable complex abelian variety (which include any abelian variety of dimension atmost 5, powers of complex elliptic curves, etc.) which vanishes as a morphism on a certain quotient of its middle singular cohomology, then it vanishes as a morphism on the deepest part of a particular filtration on the Chow group of 0-cycles of the abelian variety. As a consequence, we prove that given an automorphism of such an abelian variety, which acts as Identity on a certain quotient of its middle singular cohomology, then it acts as Identity on the deepest part of this filtration on the Chow group of 0-cycles of the abelian variety. As an application, we prove that for the Generalized Kummer variety associated to a complex abelian surface and the automorphism induced from a symplectic automorphism of the complex abelian surface, the automorphism of the Generalized Kummer variety acts as Identity on its Chow group of 0-cycles.

1 Introduction

Given a smooth projective variety X over an algebraically closed field $k \subset \mathbf{C}$, we can associate two important invariants, the Chow groups and singular cohomology groups, both indexed by non-negative integers. These two invariants are related by cycle class homomorphisms from Chow groups to even degree cohomology groups. Further, the kernel of the cycle class map is related to a Hodge structure on the singular cohomology groups by the Abel-Jacobi map. D. Mumford showed in [21] that, if X is a complex surface which admits a non-zero holomorphic 2-form (i.e. $H^0(X, \Omega_X^2) \neq 0$), then $CH_0(X)$ (the Chow group of 0-dimensional cycles) is “infinite dimensional”. This result suggested that the singular cohomology groups, or rather the Hodge structure on the cohomology groups, dictates the structure of the Chow groups.

Conjectural formulation of such a relation was first initiated by S. Bloch (See [5, Conjecture 1.8]). This has been vastly generalized into a finer conjecture known as the Bloch-Beilinson conjecture [16], which says that

Conjecture 1 (Bloch-Beilinson). *If X is a smooth projective variety of dimension d over \mathbf{C} , then for each $k \geq 0$, there exists a decreasing filtration $G^i CH^k(X)$ on the Chow groups with rational coefficients satisfying:*

1. (Finiteness) $G^{k+1} CH^k(X) = 0$

2. (Functoriality) The filtration G^i is stable under correspondences: if $\Gamma \in CH^l(X \times Y)$ and Y is a smooth projective variety over \mathbf{C} , then the maps

$$\Gamma_* : CH^k(X) \rightarrow CH^{k+l-\dim X}(Y)$$

satisfy

$$\Gamma_* G^i CH^k(X) \subset G^i CH^{k+l-\dim X}(Y).$$

3. The induced map

$$gr_G^i \Gamma_* : gr_G^i CH^k(X) \rightarrow gr_G^i \Gamma_* : gr_G^i CH^{k+l-\dim X}(Y)$$

vanishes if the map

$$[\Gamma]_* : H^{2k-i}(X, \mathbf{Q}) \rightarrow H^{2k+2l-2n-i}(Y, \mathbf{Q})$$

vanishes on $H^{r,s}(X)$ for $s \leq k - i$.

In particular, for $k = d = \dim X$, the conjecture predicts for the Chow group of 0-cycles with \mathbf{Q} -coefficients $CH^d(X) = CH_0(X)$ the following:

Conjecture 2. *There exists a filtration $G^i CH_0(X)$ such that $G^{d+1} CH_0(X) = 0$, stable under suitable correspondences and satisfying:*

For a correspondence $\Gamma \in CH^{\dim Y}(X \times Y)$, the map

$$gr_G^i \Gamma_* : gr_G^i CH_0(X) \rightarrow gr_G^i \Gamma_* : gr_G^i CH_0(Y)$$

is 0 if

$$[\Gamma]_* : H^0(X, \Omega_X^i) \rightarrow H^0(Y, \Omega_Y^i)$$

is 0.

Our goal here is to study a particular natural filtration G^\bullet on $CH_0(X)$ for X an abelian variety over an algebraically closed field k of char 0. This filtration was first studied by Bloch in [4].

Definition 1. *Let $Pic(X)$ (and $Pic^0(X)$) be the group of divisors (and divisors algebraically equivalent to 0, resp.) modulo rational equivalence on X . Now the intersection of divisors gives a decreasing filtration G^\bullet on the Chow group of 0-cycles, $CH_0(X)$ as follows:*

$$G^i := Pic^0(X)^{\cap i} \cap Pic(X)^{\cap (d-i)}, \text{ for } i \geq 0$$

where \cap is the intersection product on cycles on X .

One expects that, this filtration satisfies the Bloch-Beilinson conjecture 2 for Chow group of X with rational coefficients. In particular, for $Y = X$, $\Gamma = \Gamma_f - \Delta$, where Γ_f is the graph of f and Δ is the diagonal of X , the Conjecture 2 predicts the following (we will be using homological notation $H_d(X) := H_d(X, \mathbf{Q})$, $H_{d,0}(X) = H^0(X, \Omega_X^d)$ as will be made clear later):

Conjecture 3. *Let X be an abelian variety of dimension d over \mathbf{C} . Suppose $f : X \rightarrow X$ is an automorphism of X such that the induced morphism $f_* : H_{d,0}(X) \rightarrow H_{d,0}(X)$ is the Identity (abbreviated as Id). Then $f_* = Id : G^d \rightarrow G^d$ induced by the restriction of $f_* : CH_0(X) \rightarrow CH_0(X)$.*

This conjecture is difficult to answer in general. We will reformulate this conjecture to Conjecture 4 which is more tractable in view of the recent results of C. Vial [27]. For this let us consider more closely the condition $f_* : H_{d,0}(X) \rightarrow H_{d,0}(X)$ is the Identity. For this discussion, X can be any smooth projective variety of dimension d over \mathbf{C} . Let $\Gamma_f :=$ the graph of $f \in CH^d(X \times X)$. Then $[\Delta_X - \Gamma_f]_*$ is a \mathbf{Q} -linear map on $H_d(X)$ which after tensoring with \mathbf{C} is 0 on $H_{d,0}(X)$. Note that $[\Delta_X - \Gamma_f]_* : H_d(X) \rightarrow H_d(X)$ is a morphism of \mathbf{Q} -Hodge structures, hence $\ker([\Delta_X - \Gamma_f]_*)$ is a \mathbf{Q} -sub Hodge structure of $H_d(X)$, which contains $H_{d,0}(X)$ after tensoring with \mathbf{C} . Therefore, $\ker([\Delta_X - \Gamma_f]_*)$ contains the smallest \mathbf{Q} -sub Hodge structure of $H_d(X)$ which contains $H_{d,0}(X)$ after tensoring with \mathbf{C} , which we will write as $H_d(X)_{tr}$ [for transcendental cohomology]. Thus, we can replace the assumption in the Conjecture 3 by the condition that $f_* : H_d(X)_{tr} \rightarrow H_d(X)_{tr}$ is the Identity.

In terms of the **Coniveau filtration** $N^\bullet H_d(X, \mathbf{Q})$ on $H_d(X)$ defined in Definition 2.1.1 and the Hodge filtration $F^p H_d(X, \mathbf{C})$ on $H_d(X)$,

$$N^p H_d(X) \subseteq F^p H_d(X, \mathbf{C}) \cap H_d(X), \text{ for } p \geq 0.$$

We recall the Grothendieck's Generalized Hodge Conjecture here. We say that X satisfies **GHC**(X, p, d), if the maximal \mathbf{Q} -Hodge structure contained in $F^p H_d(X, \mathbf{C}) \cap H_d(X)$ is $N^p H_d(X)$.

Fixing a polarization on $H_d(X)$, we have the orthogonal complement $H_d(X)_{tr}^\perp$ of $H_d(X)_{tr}$ in $H_d(X)$, which is a \mathbf{Q} -sub Hodge structure contained in $F^1 H_d(X, \mathbf{C}) \cap H_d(X)$. By earlier discussion, $H_d(X)_{tr}$ is contained in $N^1 H_d(X)^\perp$ as $N^1 H_d(X)^\perp$ is \mathbf{Q} -sub Hodge structure containing $H_{d,0}(X)$ in the Hodge decomposition. Hence $N^1 H_d(X) \subseteq H_d(X)_{tr}^\perp$. Thus, we have a surjective homomorphism

$$\frac{H_d(X)}{N^1 H_d(X)} \twoheadrightarrow \frac{H_d(X)}{H_d(X)_{tr}^\perp} = H_d(X)_{tr}$$

which will be an isomorphism assuming **GHC**($X, 1, d$). Examples of this situation will be listed in the Section 2.4.

Next, we will define **Niveau filtration** \tilde{N}^\bullet on the homology of X in the next section (see Definition 2.1.2).

Using the properties of Niveau and Coniveau filtration, we have

$$\frac{H_d(X)}{\tilde{N}^1 H_d(X)} \twoheadrightarrow \frac{H_d(X)}{N^1 H_d(X)} \twoheadrightarrow \frac{H_d(X)}{H_d(X)_{tr}^\perp} = H_d(X)_{tr}.$$

With this in hindsight, we will modify the Conjecture 3, as follows:

Conjecture 4. *Let X be an abelian variety of dimension d over \mathbf{C} . Suppose $f : X \rightarrow X$ is an automorphism of X such that the induced morphism $f_* : \frac{H_d(X)}{\tilde{N}^1 H_d(X)} \rightarrow \frac{H_d(X)}{\tilde{N}^1 H_d(X)}$ is the Id. Then $f_* = \text{Id} : G^d \rightarrow G^d$ induced by the restriction of $f_* : CH_0(X) \rightarrow CH_0(X)$.*

By the discussion above under **GHC**, the assumption on f in Conjecture 4 implies the assumption on f in Conjecture 3.

Statements of the results In this paper, we propose to answer Conjecture 3 and 4 for suitable abelian varieties X , which follows from the following more general theorem. The main theorem in this paper is:

Theorem 1 (See Theorem 2.6.1). *Let X be a complex abelian variety satisfying the assumption $(*)$ as in the Section 2.3. Suppose $\Gamma \in CH^d(X \times X)$ is a correspondence from X to X which is symmetrically distinguished. Let $[\Gamma]_* : Gr_{\tilde{N}^\bullet}^0 H_d(X) \rightarrow Gr_{\tilde{N}^\bullet}^0 H_d(X)$ be the induced morphism which is 0 on the 0th graded piece with respect to*

the Niveau filtration on the homology. Then $[\Gamma]_* = 0 : \cap_{i=1}^d \text{Pic}^0(X) \rightarrow \cap_{i=1}^d \text{Pic}^0(X)$ induced by the restriction of $[\Gamma]_* : CH_0(X) \rightarrow CH_0(X)$.

We note that the assumption $(*)$ in the Theorem 1 holds for many abelian varieties, e.g. listed in the Section 2.4.

As a consequence, we prove,

Theorem 2 (See Corollary 2.6.3). *Let X be a complex abelian variety of dimension d satisfying the assumption $(*)$ as in the Section 2.3. Then X satisfies the Conjecture 4.*

Theorem 2 can be applied to many abelian varieties listed in the Section 2.4, including of dimension atmost 5, product of elliptic curves, etc.

We deduce Conjecture 3 from Theorem 2, for suitable abelian varieties.

Theorem 3. *Let X be a complex abelian variety of dimension d satisfying the assumption $(*)$ as above, $N^1 H_d(X) = \tilde{N}^1 H_d(X)$ and furthermore $\text{GHC}(X, 1, d)$ holds true. Then X satisfies the Conjecture 3.*

Theorem 3 follows from Theorem 2 and the discussion following the Conjecture 3. We note that the Theorem 3 can be applied to abelian varieties listed in the Section 2.4, for which the assumption $\text{GHC}(X, 1, d)$ holds true, which include among others:

1. abelian varieties of dimension ≤ 2 ,
2. abelian threefolds for which $\text{GHC}(X, 1, 3)$ holds true,
3. product of elliptic curves.

Finally, we discuss various applications of Theorem 2. We consider the Kummer surfaces and in general *Generalized Kummer varieties* and their Chow groups of 0-cycles. We first discuss, the structure of the Chow group of 0-cycles with \mathbf{Q} -coefficients, which follows from the Lemma 3.2.6 and the Remark 3.2.7. As a consequence, we prove the following proposition:

Proposition 1 (see Proposition 3.2.5). *Let X be a complex abelian surface. Let K_n be the n^{th} Generalized Kummer variety associated to X . Suppose f_n is the automorphism of K_n induced from an automorphism f of X such that f is symplectic on X (i.e. which acts as Id on $H_{2,0}(X)$). Then f_n is symplectic and f_n acts as Id on $CH_0(K_n)$.*

Throughout the paper, we will be using the following notations:

Notations X will denote a smooth projective variety of pure dimension d over an algebraically closed field k , a subfield of the field of complex numbers \mathbf{C} (In particular of char 0), unless otherwise specified. Set $H_i(X) = H_i(X(\mathbf{C}), \mathbf{Q})$ which is isomorphic to $H^{2d-i}(X(\mathbf{C}), \mathbf{Q})$ by Poincare duality. $H_{i,j}(X) := H^{i,j}(X) = H^j(X, \Omega_X^i)$ the Hodge cohomology groups associated to the Hodge structure on $H_{i+j}(X)$. Denote $F^p H_d(X, \mathbf{C}) = \oplus_{a \geq p} H^{a, d-a}(X)$, the Hodge filtration on $H_d(X)$. The Chow group of X with \mathbf{Q} coefficients is denoted by $CH^i(X)$. $CH_{num}^i(X)$ is $CH^i(X)$ modulo numerical equivalence.

Structure of the paper: We define Coniveau and Niveau filtrations of the homology in Definitions 2.1.1 and 2.1.2. We recall the notion of Chow-Künneth decomposition of a diagonal and some results for abelian varieties in Section 2.2. We recall the main results of [27], Theorem 2.3.1 and Theorem 2.3.3 and the definition of symmetrically distinguished cycles in Definition 2.5.1. Using these we prove the main result of this paper in Theorem 2.6.1. From this we deduce Theorem 2. In Section 3, we discuss some applications of this result, especially the Lemma 3.2.6 and Proposition 1.

2 Main Theorem

2.1 Filtrations on Homology and their Properties

The coniveau filtration on the homology groups has been studied by many authors including Bloch-Ogus [7], Jannsen [16]. There is yet another filtration called Niveau filtration which was implicit in [25]. Also the Niveau filtration was studied along with other filtrations on the homology by Friedlander-Mazur [10], where it was called Correspondence filtration. Here we recall their definitions, following the exposition in [27].

Definition 2.1.1. The **Coniveau (arithmetic) filtration** on homology $H_i(X)$ is given by

$$N^j H_i(X) = \sum \text{Im}(\Gamma_* : H^{i-2j'}(W) \rightarrow H_i(X))$$

where the sum ranges over all integers $j' \geq j$, all smooth projective varieties W and over all correspondences $\Gamma \in CH_{i-j'}(W \times X)$.

By considering $\mathbf{P}^{j'-j} \times W$ instead of W and $\mathbf{P}^{j'-j} \times \Gamma$ instead of Γ we see that it is superfluous to consider integers $j' > j$. Therefore

$$N^j H_i(X) = \sum \text{Im}(\Gamma_* : H^{i-2j}(W) \rightarrow H_i(X))$$

where the sum ranges over all smooth projective varieties W and over all correspondences $\Gamma \in CH_{i-j}(W \times X)$. Further by resolution of singularities we see that

$$N^j H_i(X) = \sum \text{Im}(f_* : H^{i-2j}(W) \rightarrow H_i(X))$$

where the sum ranges over all morphisms $f : W \rightarrow X$ from a smooth projective variety W of pure dimension $\leq i - j$ to X .

Definition 2.1.2. The **correspondence filtration** or **Niveau** on $H_i(X)$ can be defined as

$$\tilde{N}^j H_i(X) := \sum \text{Im}(\Gamma_* : H_{i-2j'}(W) \rightarrow H_i(X))$$

where the sum ranges over all integers $j' \geq j$, all smooth projective varieties W and over all correspondences $\Gamma \in CH_{i-j'}(W \times X)$.

Actually the condition $j' \geq j$ and flexibility over dimension of W is not needed as if $j' > j$ then replace W by $\mathbf{P}^{j'-j} \times W$ and Γ by $0 \times \Gamma \in CH^{d-j}(\mathbf{P}^{j'-j} \times W \times X)$. Thus can take $j' = j$ in the definition of niveau filtration. Further is also possible to restrict to W of dimension $i - 2j$. If $\dim W > i - 2j$, then any smooth linear section $u : W' \rightarrow W$ of dimension $i - 2j$ induces a surjection $H_i - 2j(W') \rightarrow H_{i-2j}(W)$. Replace W by

W' and Γ by $\Gamma \circ u$. If $\dim W < i - 2j$, then replace W by $\mathbf{P}^m \times W$ where $m = i - 2j - \dim W$ and Γ by $\mathbf{P}^m \times \Gamma \in CH^{d-j}(\mathbf{P}^m \times W \times X)$. Therefore we have

$$\tilde{N}^j H_i(X) := \sum_{\Gamma \in CH^{d-j}(W \times X)} \text{Im}(\Gamma_* : H_{i-2j}(W) \rightarrow H_i(X))$$

where the sum runs over all smooth projective varieties W of dimension $i - 2j$ and all correspondences $\Gamma \in CH_{i-j}(W \times X) = CH^{d-j}(W \times X)$. Because $H_i(X)$ is a finite dimensional \mathbf{Q} -vector space, we can find finitely many W_s such that Γ_s is a correspondence from W_s to X and take $W_{i,j} = W_1 \sqcup \dots \sqcup W_r$ and $\Gamma_{i,j} = \sqcup_s (\Gamma_s)$ satisfying $W_{i,j}$ is a smooth projective variety of pure dimension $i - 2j$ and the correspondence $\Gamma_{i,j} \in CH_{i-j}(W_{i,j} \times X)$ such that $\tilde{N}^j H_i(X) = (\Gamma_{i,j})_* H_{i-2j}(W_{i,j})$.

These two filtrations are related to each other as follows:

1. N^\bullet and \tilde{N}^\bullet are decreasing filtrations such that

$$\begin{aligned} H_i(X) &= N^0 H_i(X) \supseteq N^1 H_i(X) \supseteq \dots \supseteq N^{\lfloor i/2 \rfloor} H_i(X) \supseteq N^{\lfloor i/2 \rfloor + 1} H_i(X) = 0 \text{ and} \\ H_i(X) &= \tilde{N}^0 H_i(X) \supseteq \tilde{N}^1 H_i(X) \supseteq \dots \supseteq \tilde{N}^{\lfloor i/2 \rfloor} H_i(X) \supseteq \tilde{N}^{\lfloor i/2 \rfloor + 1} H_i(X) = 0 \end{aligned}$$

2. By the modified characterization of the filtrations we have an inclusion $\tilde{N}^j \subseteq N^j$. By Lefschetz theorem the cup product by the cohomology class by an ample line bundle on W gives an isomorphism of $H_i(W) \rightarrow H^i(W)$ for $i = 0, 1$. Hence $\tilde{N}^{\lfloor i/2 \rfloor} H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$. More generally, it was conjectured by Friedlander and Mazur [10, p. 71] that these two filtrations agree for $k = \mathbf{C}$. Vial [27, Proposition 1.1] has shown that, if the Lefschetz standard conjecture holds for all smooth projective varieties of dimension $< \dim X$, then the two filtrations agree.

2.2 Chow-Künneth Decomposition

Next we recall some terminology from the theory of Chow motives.

Definition 2.2.1. X of dimension d is said to satisfy a Chow-Künneth decomposition, if there is a collection $\{\pi_i\}_{i=0}^{2d}$ of codimension d cycles on $X \times X$ such that the following relations hold in $CH_d(X \times X)$.

1. $\Delta_X = \sum_{i=0}^{2d} \pi_i$,
2. ${}^t \pi_i = \pi_{2d-i}$, (duality)
3. $\pi_i \circ \pi_i = \pi_i$, $\pi_i \circ \pi_j = 0$ if $i \neq j$, (idempotent, mutually orthogonal resp.) and
4. π_i acts on $H_j(X, \mathbf{Q})$ as $\delta_{ij} \text{Id}_{H_j(X, \mathbf{Q})}$.

Some examples of smooth projective varieties admitting a Chow-Künneth decomposition are curves, surfaces. For abelian varieties X , we have the multiplication map by $n : X \rightarrow X$ for $n \in \mathbf{Z}^*$, which induces $n_* : CH^i(X) \rightarrow CH^i(X)$. By Beauville's theorem:

Theorem 2.2.2 (Beauville [3]). *For X an abelian variety of dimension d , there is a decomposition of $CH^i(X)$ given by*

$$CH^i(X) = \bigoplus_{s=i-d}^i CH^i(X)_s,$$

where

$$CH^i(X)_s = \{\alpha \in CH^i(X) \mid n_*\alpha = n^{2d-2i+s}\alpha \text{ for all } n \in \mathbf{Z}^*\}.$$

We further have by Shermenev [26], Denninger-Murre [9],

Theorem 2.2.3. *For X an abelian variety of dimension d over $k = \bar{k}$,*

1. *X has a Chow-Künneth decomposition*
2. *There exist Chow-Künneth components π_i such that $n_* \circ \pi_i = \pi_i \circ n_* = n^{2d-2i+s}\pi_i$ for all n and such π_i 's are unique.*

Combining this with Beauville's result, we get that

$$CH^i(X)_s = (\pi_{2d-2i+s})_* CH^i(X). \quad (1)$$

2.3 Refined Chow-Künneth Decomposition

In this section, we will consider X to be a smooth projective variety over an algebraically closed field k and will concentrate on $i = d$, $H_d(X, \mathbf{Q})$ the middle singular homology of X . Given a Chow-Künneth decomposition of X , under some further assumptions (*) and (**) defined below, we follow C. Vial's recipe in [27] to produce a refined decomposition of the cycle π_d into mutually orthogonal, idempotents given by

$$\pi_d = \sum_{j=0}^{\lfloor d/2 \rfloor} \pi_{d,j}$$

such that $(\pi_{d,j})_* H_*(X, \mathbf{Q}) = Gr_{\tilde{N}^\bullet}^j H_d(X)$.

Let us setup some assumptions. Let L be an ample line bundle on X which defines an embedding $X \subseteq \mathbf{P}_k^N$ and hence for any integer $i \leq d$, a map $L^{d-i} : H_{2d-i}(X) \rightarrow H_i(X)$ given by intersecting $d-i$ times with the cohomology class of a hyperplane section of H of X . This map is induced by a correspondence on $X \times X$ given by $X \times H$ and is an isomorphism of Hodge structures (Hard Lefschetz theorem). We say X satisfies the property B_i if the isomorphism $L^{i-d} := (L^{d-i})^{-1} : H_i(X) \rightarrow H_{2d-i}(X)$ is induced by a correspondence. If X satisfies property B_i for every i then we say X satisfies property **B**. The property **B** is satisfied by abelian varieties, curves, surfaces, complete intersections and any products, hypersurface intersections or their finite quotient. One of Grothendieck's conjecture says that all smooth projective varieties satisfy the property **B**.

Assumption (*):

1. X satisfies **B** and
2. for all $i \leq d$ and all $j \geq 1$, either there exists $W_{d,j}$ as before satisfying B_l for all $l \leq d-2j-2$ or $N^{j+1}H_d(X) = \tilde{N}^{j+1}H_d(X)$.

As the property B_1 holds for all smooth projective varieties, the property (*) holds for all smooth projective varieties of dimension at most 5 that satisfy property **B**, as the $W_{d,j}$ need to only satisfy B_l for $l = 1$. In particular property (*) holds for curves, surfaces, abelian varieties of dimension ≤ 5 , complete intersections of dimension ≤ 5 , uniruled 3-folds, rationally connected 5-folds with $H^3(X, \Omega_X) = 0$.

Further, consider varieties X satisfying the

Assumption (**): $\text{Ker}(cl : CH_d(X \times X) \rightarrow H_{2d}(X \times X))$ is a nilpotent ideal.

Here $CH_d(X \times X)$ is a ring with respect to the multiplication by composition of correspondences denoted by \circ . For example, varieties dominated by a product of curves satisfy (**). These include abelian varieties and Fermat varieties. More generally any variety satisfying Kimura's finite dimensionality as in [17] satisfies (**).

We recall the Theorem 1.1 from C. Vial's paper [27], restated here for the applications we have in mind. We add the proof here for the sake of completeness.

Theorem 2.3.1. *If X satisfies (*) then for each integer $j \geq 0$, there exists a cycle $\pi_{d,j}$ inducing the orthogonal projection $H_d(X) \rightarrow Gr_{\tilde{N}^\bullet}^j H_d(X) \rightarrow H_d(X)$.*

Proof. Fix $j \geq 1$. Let W be a smooth projective variety of dimension $d - 2j$ such that $\text{Im}(\Gamma_* : H_{d-2j}(W) \rightarrow H_d(X)) = \tilde{N}^j H_d(X)$ and $\Gamma \in CH_{d-j}(Z \times X)$. Take π_d which induces the projector $H_*(X) \rightarrow H_d(X) \rightarrow H_*(X)$. Then $\pi_d \circ \Gamma$ induces $H_*(W) \rightarrow H_*(X)$ such that $\text{Im}([\pi_d \circ \Gamma]_* : H_{d-2j}(W) \rightarrow H_d(X)) = \tilde{N}^j H_d(X)$. Note that $[\Gamma]_*$ on $H_k(W)$ is 0 if $k \neq d - 2j$. Suppose we have constructed the cycles $\pi_{d,r}$ for $r > j$, then we will construct the cycles $\pi_{d,j}$ as asserted. For $r > [d/2]$ set $\pi_{d,r} = 0$.

Consider the cycle $\gamma := (\pi_d - \sum_{r>j} \pi_{d,r}) \circ \Gamma$.

The induced morphism on homology $\gamma_* : H_*(W) \rightarrow H_*(X)$ can be identified with $Gr_{\tilde{N}^\bullet}^j H_d(X)$, the orthogonal complement of $\tilde{N}^{j+1} H_d(X)$ in $\tilde{N}^j H_d(X)$, with respect to the pairing on $H_d(X)$.

Then transpose of γ induces the morphism

$$[{}^t\gamma]_* : H_d(X) \rightarrow H_{d-2j}(W).$$

Consider now the Lefschetz primitive decomposition of W give by

$$H_{d-2j}(W) = \bigoplus_{m \geq 0} L_W^m H_{d-2j+2m}(W)_{\text{prim}}$$

and write p_m for the algebraic correspondence which induces the projection

$$[p_m]_* : H_{d-2j}(W) \rightarrow L_W^m H_{d-2j+2m}(W)_{\text{prim}}.$$

Case 1: $N^{j+1} H_d(X) = \tilde{N}^{j+1} H_d(X)$: First we observe that $[\gamma \circ p_m]_* = 0$ for $m > 0$.

If $x \in L_W^m H_{d-2j+2m}(W)_{\text{prim}}$, then we have $\gamma_*(x) = \gamma_*((p_m)_*(y))$ for some y in the primitive cohomology $H_{d-2j+2m}(W)_{\text{prim}} = H^{d-2j-2m}(W)_{\text{prim}} \subset H^{d-2j-2m}(W)$, so that for the correspondence $\gamma \circ p_m \in CH_{d-j-m}(W \times X)$ such that $\gamma_*(x) = [\gamma \circ p_m]_*(y) \in N^{j+m} H_d(X)$.

Thus if $m > 0$, then $\gamma_*(x) \in N^{j+m} H_d(X) \subseteq N^{j+1} H_d(X) = \tilde{N}^{j+1} H_d(X)$. But recall the image of γ_* is $Gr_{\tilde{N}^\bullet}^j H_d(X)$, hence $\gamma_*(x) = 0$. Therefore the map $[\gamma]_* : H_*(W) \rightarrow H_*(X)$ factors through $H_{d-2j}(W)_{\text{prim}}$.

Define $\phi := \gamma \circ {}^t\gamma$.

Case 2: W satisfies the property B_l for all $l \leq d - 2j - 2$.

The morphism $s_W := \sum (-1)^r p_r$ on $H_{d-2j}(W)$ is induced by a correspondence by [8, Lemma 7] and the pairing $(-, s_W(-))$ defines a polarization on $H_{d-2j}(W)$. Define $\phi := \gamma \circ s_W \circ {}^t\gamma$.

Either way, by [29, Lemma 5] we have

$\phi_* H_*(X) = \gamma_* H_*(W) = Gr_{\tilde{N}^\bullet}^j H_d(X)$. Therefore, ϕ_* restricts to an isomorphism on the finite dimensional

\mathbf{Q} -vector space $Gr_{\tilde{N}\bullet}^j H_d(X)$. By the Cayley-Hamilton theorem, there is a polynomial $P \in \mathbf{Q}[T]$ such that $(\phi_{Gr_{\tilde{N}\bullet}^j H_d(X)})^{-1} = P(\phi_{Gr_{\tilde{N}\bullet}^j H_d(X)})$. Then let $\psi = P(\phi)$. Define $\pi_{d,j} := \psi \circ \phi$.

One can check that, $\pi_{d,j}$ induces Id on the $Gr_{\tilde{N}\bullet}^j H_d(X)$ and 0 on its orthogonal complement under the polarization chosen and also $[\pi_{d,j}]_* \circ [\pi_{d,j}]^* = [\pi_{d,j}]_*$. Next define $\pi_{d,0} := \pi_d - \sum_{j \geq 1} \pi_{d,j}$. Hence the theorem is proved. \square

Let us recall a lifting lemma:

Lemma 2.3.2 (Kleiman/Jannsen). *Let X be a smooth projective variety satisfying the property $(**)$ as stated above. Let $c_1, \dots, c_n \in CH^d(X \times X)$ be correspondences such that $cl(c_i) \in H^*(X \times X)$ define mutually orthogonal idempotents adding to identity. Then there exists mutually orthogonal idempotents $p_1, \dots, p_n \in CH^d(X \times X)$ adding to identity such that $cl(p_i) = cl(c_i)$ for all i . Moreover any two such choices $\{p_1, \dots, p_n\}$ and $\{p'_1, \dots, p'_n\}$ are conjugate by an element lying above the identity i.e. there exists a nilpotent correspondence $\eta \in CH^d(X \times X)$ such that $p_i = (1 + \eta) \circ p_i \circ (1 + \eta)^{-1}$ for all i .*

We recall the Theorem 1.2. from C. Vial's paper [27].

Theorem 2.3.3. [27, Theorem 1.2] *Let X be a smooth projective variety for which the properties $(*)$ and $(**)$ hold as defined above. Then there exists a collection of codimension d cycles $\{\Pi_{d,j}\}_{j=0}^{\lfloor d/2 \rfloor}$ on $X \times X$ which are mutually orthogonal, idempotent, and for each j , $\Pi_{d,j}$ is homologically equivalent to $\pi_{d,j}$ such that $\pi_d = \sum_j \Pi_{d,j}$ and $H_*(X, \Pi_{d,j}) = Gr_{\tilde{N}\bullet}^j H_d(X)$. For any such choice of idempotents we have:*

1. $\Pi_{d,j}$ acts as 0 on $CH_l(X)$ if either $l < j$ or $l > d - j$.
2. $\Pi_{d,j}$ acts as 0 on $CH_l(X)$ if either $l = d - j$ and $d < 2l$.
3. $\Pi_{d,j}$ acts as 0 on $CH_l(X)$ if either $l + 1 = i - j$ and $d \leq 2l$.

2.4 A List of Complex Abelian Varieties and the Assumption $(*)$

We will consider the following list of complex abelian varieties to which we can apply the Theorem 2.3.3 i.e. for which the assumption $(*)$ holds true (see Remark 2.4.2).

A List:

1. An abelian variety X of dimension ≤ 5 .
2. An abelian variety X for which the Hodge group (denoted by $Hg(X)$) is equal to the symplectic group on the vector space $H_1(X, \mathbf{Q})$ with polarization β , denoted by $Sp(H_1(X, \mathbf{Q}), \beta)$.

Next we list abelian varieties X , for which the Generalized Hodge Conjecture is known for all powers of X (see [14] and the references therein).

For a complex abelian variety X , let $D(X)$ = the endomorphism algebra $End(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. We say

- (a) X is of type I if $D(X)$ is a totally real field and
- (b) X is of type II if $D(X)$ is a totally indefinite quaternion algebra over a totally real number field.

3. An abelian variety X such that the Hodge ring of X^n is generated by divisors on X^n for all $n \geq 1$ (known as stably non-degenerate), and all of whose simple components are of type I or type II.
4. An abelian variety X such that $\dim X = \dim Hg(X)$ (called non-degenerate), of CM type with CM field $E = D(X)$ so that, E is an imaginary quadratic field over a totally real field F of degree d over \mathbf{Q} , such that the degree $[\bar{E} : \bar{F}] = 2^d$, where bars denote the Galois closure of respective fields. ([2, Example 1]).
5. A product of complex elliptic curves.

Theorem 2.4.1. *Let X be an abelian variety in the cases 2, 3, 4, 5 defined above. Then for all j and i ,*

$$\tilde{N}^j H_i(X) = N^j H_i(X).$$

Proof. • X is in Case 2: The proof follows from [11, Proposition 4.4] and [10, Theorem 7.3] after one observes that in loc. cit. $\tilde{N}^j H_i(X) = C_j H_i(X)$ and $N^j H_i(X) = G_j H_i(X)$.

- X in Case 3: For the proof, we refer to [1, Theorem 8.1].
- X is in Case 4: The proof follows from [2, Example 1], where it is proved that such X is dominated by powers of X and since X is non-degenerate (Ribet [23, Corollary 3.6, p. 87]), GHC holds for all powers of X . Now the conclusion follows from [1, Proposition 5.1 and Theorem 5.2].
- X is in Case 5: The proof follows from [18, p. 195].

□

Remark 2.4.2. 1. As mentioned earlier, X in case 1 satisfies the assumption (*).

2. If X belongs to the cases 2, 3, 4, 5, then X satisfies the assumption (*), as it follows from the above Theorem 2.4.1 for $i = d = \dim X$.

Applying the Theorem 2.3.3 to the abelian varieties in the List above we get:

Corollary 2.4.3. *Let X be a complex abelian variety in the List above. Then there exists a collection of cycles $\{\Pi_{d,j}\}_{j=0}^{\lfloor d/2 \rfloor}$ on $X \times X$ of codimension d which are mutually orthogonal, idempotent, such that $\pi_d = \sum_{j=0}^{\lfloor d/2 \rfloor} \Pi_{d,j}$ and $H_*(X, \Pi_{d,j}) = Gr_{\tilde{N}}^j H_d(X)$. Moreover, the idempotents satisfy:*

1. $\Pi_{d,j}$ acts as 0 on $CH_l(X)$ if either $l < j$ or $l > d - j$.
2. $\Pi_{d,j}$ acts as 0 on $CH_l(X)$ if either $l = d - j$ and $d < 2l$.
3. $\Pi_{d,j}$ acts as 0 on $CH_l(X)$ if either $l + 1 = i - j$ and $d \leq 2l$.

We will further need a notion of symmetrically distinguished cycles on abelian varieties for our Theorem 2.6.1.

2.5 Symmetrically Distinguished Cycles on Abelian Varieties

Throughout this section, X will denote an abelian variety over k . The notion of symmetrically distinguished cycles was first introduced by P. O'Sullivan in [22, Definition 6.2.1].

Definition 2.5.1. Let α be an element in the i^{th} Chow group $CH^i(X)$ of X . Let $V_m(\alpha)$ be the \mathbf{Q} -subspace of $CH(X^m)$ generated by elements of the form

$$p_*(\alpha^{r_1} \otimes \alpha^{r_2} \otimes \dots \otimes \alpha^{r_n})$$

where $n \leq m$, the r_j are the integers ≥ 0 , and $p : X^m \rightarrow X^n$ is a closed immersion with each component $p : X^m \rightarrow X$ either a projection or the composite of a projection with $(-1)_X : X \rightarrow X$. Then α will be called the **symmetrically distinguished** element if for every m the restriction of the quotient $CH(X^m) \rightarrow CH_{num}(X^m)$ restricted to $V_m(\alpha)$ is injective. An arbitrary element of $CH^*(X)$ will be called **symmetrically distinguished** if each of its homogeneous component is symmetrically distinguished.

The main result of O’Sullivan’s paper [22, Corollary 6.2.6] is the following theorem:

- Theorem 2.5.2.** 1. *For every cycle in $CH_{num}(X)$ there exists a unique symmetrically distinguished cycle in $CH(X)$ which lies above it.*
2. *The symmetrically distinguished cycles in $CH(X)$ form a \mathbf{Q} -vector subspace, and the product of symmetrically distinguished cycles in $CH^i(X)$ and $CH^j(X)$ is a symmetrically distinguished cycle in $CH^{i+j}(X)$.*
3. *For any homomorphism of abelian varieties $f : X \rightarrow Y$, the pullback f^* and push forward f_* preserve the symmetrically distinguished cycles.*

- Remark 2.5.3.** 1. From this theorem, we can deduce that if $f : X \rightarrow X$ is an automorphism of abelian variety X , then the graph of f denoted by Γ_f , as an element in $CH^d(X \times X)$ is a symmetrically distinguished. Indeed the class $[X] \in CH^0(X)$ is a symmetrically distinguished element and $[\Gamma_f] = (1_X \times f)_*([X])$. Thus the claim follows from Theorem 2.5.2 (3). We make this remark here, separately for future reference. Note that in the above remark, it is enough to assume f is any homomorphism of abelian varieties, not necessarily an automorphism.
2. Another important example of a symmetrically distinguished cycle on X is a symmetric divisor D on X i.e. $D \in CH^1(X)$ such that $(-1)_X^* D = D$ in $CH^1(X)$. Hence, cycles on X generated by symmetric divisors on X are symmetrically distinguished as it follows from Theorem 2.5.2 (2).

2.6 Main Theorem for Abelian Varieties

The **main theorem** of this paper is the following:

Theorem 2.6.1. *Let X be a complex abelian variety satisfying the assumption (*). Suppose $\Gamma \in CH^d(X \times X)$ is a correspondence from X to X which is symmetrically distinguished. Let $[\Gamma]_* : Gr_{N\bullet}^0 H_d(X) \rightarrow Gr_{N\bullet}^0 H_d(X)$ be the induced morphism which is 0 on the 0th graded piece with respect to the Niveau filtration on the homology. Then $[\Gamma]_* = 0 : \cap_{i=1}^d Pic^0(X) \rightarrow \cap_{i=1}^d Pic^0(X)$ induced by the restriction of $[\Gamma]_* : CH_0(X) \rightarrow CH_0(X)$.*

The proof of Theorem 2.6.1 will follow from the following proposition.

Proposition 2.6.2. *Let X be a complex abelian variety satisfying the assumption (*). Then Theorem 2.3.3 holds for X and further there exist $\Phi_{d,j}$ symmetrically distinguished in the sense of Definition 2.5.1 such that $\Phi_{d,j} \in CH^d(X \times X)$ are mutually orthogonal, idempotents and $\pi_d = \sum_j \Phi_{d,j} \in CH^d(X \times X)$ and the properties 1 to 3 of Corollary 2.4.3 hold for these modified $\Phi_{d,j}$ ’s when replaced by $\Pi_{d,j}$.*

Proof. By applying Theorem 2.5.2 (1) for $X \times X$, can lift the elements $\Pi_{d,j}$ as in Theorem 2.3.3 modulo homological equivalence (recall that homological equivalence is equivalent to numerical equivalence for abelian varieties as proved in [19]) to $CH^d(X \times X)$ such that \exists unique $\Phi_{d,j} \in CH^d(X \times X)$ which are symmetrically distinguished and $\Phi_{d,j} = \Pi_{d,j}$ in $CH_{num}^d(X \times X)$. Now let us check that the $\{\Phi_{d,j}\}$ satisfy all the properties of Theorem 2.3.3. To see that $\Phi_{d,j} \circ \Phi_{d,s} = \delta_{js}$, for Kronecker symbol, note that this equation holds modulo homological equivalence as it holds for $\Pi_{d,j}$ replaced by $\Phi_{d,j}$. Since both the cycles in the first equation are symmetrically distinguished, these are equal in $CH^d(X \times X)$. Similarly, one can see that $\Phi_{d,j}$'s are idempotents. By Lemma 2.3.2 $\{\Phi_{d,j}\}_{d,j}$ and $\{\Pi_{d,j}\}_{d,j}$ are conjugate and satisfy the properties in Corollary 2.4.3. \square

Now for the abelian variety X , we have a Chow-Künneth decomposition $\{\pi_i\}_{i=0}^{2d}$ such that $\Delta_X = \sum_i \pi_i$ by Theorem 2.2.3. By above discussion $\pi_d = \sum_j \Pi_{d,j}$ in $CH^d(X \times X)$, where $\pi_i, \Pi_{d,j}$'s are symmetrically distinguished as in Proposition 2.6.2.

Now we are all set to prove our main theorem.

Proof of the main theorem 2.6.1

Proof. Consider the following expressions in the ring $CH^d(X \times X)$ under composition of correspondences denoted by \circ .

$$\Pi_{d,0} \circ \Gamma \circ \Pi_{d,0} = (\pi_d - \sum_{j>0} \Pi_{d,j}) \circ \Gamma \circ (\pi_d - \sum_{r>0} \Pi_{d,r}) \quad (2)$$

$$= \pi_d \circ \Gamma \circ \pi_d - \sum_{j>0} \Pi_{d,j} \circ \Gamma \circ \pi_d - \sum_{r>0} \pi_d \circ \Gamma \circ \Pi_{d,r} + \sum_{j,r>0} \Pi_{d,j} \circ \Gamma \circ \Pi_{d,r} \quad (3)$$

Each of the term in the above expression induces an endomorphism of $CH_0(X)$, and by the Corollary 2.4.3(1) for $l = 0$, it follows that as endomorphisms of $CH_0(X)$:

$$[\Pi_{d,0} \circ \Gamma \circ \Pi_{d,0}]_* = [\pi_d \circ \Gamma \circ \pi_d]_* : CH_0(X) \rightarrow CH_0(X).$$

Next we show that $\Pi_{d,0} \circ \Gamma \circ \Pi_{d,0}$ is 0 in $CH^d(X \times X)$.

Consider the cycle $\Pi_{d,0} \circ \Gamma \circ \Pi_{d,0} \in CH^d(X \times X)$ modulo numerical equivalence (equivalently homological equivalence). By the assumption on Γ , $[\Gamma]_*$ is 0 on the image of $[\Pi_{d,0}]_*$, which is $Gr_{\tilde{N}}^0 H_d(X)$, by the Corollary 2.4.3 for $j = 0$. Hence it induces 0 morphism from $H_*(X) \rightarrow H_*(X)$.

Recall from the Proposition 2.6.2 and remark following it that the cycle $\Pi_{d,0} \circ \Gamma \circ \Pi_{d,0}$ is symmetrically distinguished. Hence it is numerically equivalent to 0 implies it is actually rationally equivalent to 0.

Thus, the endomorphism of $CH_0(X)$ induced by such a cycle is again 0. Hence by earlier discussion, $[\pi_d \circ \Gamma \circ \pi_d]_* : CH_0(X) \rightarrow CH_0(X)$ is 0. In other words, $[\Gamma]_* : CH_0(X) \rightarrow CH_0(X)$ is 0 on the image of $[\pi_d]_* : CH_0(X) \rightarrow CH_0(X)$. We know that $\text{Im } [\pi_d]_* : CH_0(X) \rightarrow CH_0(X) = \cap_{i=1}^d \text{Pic}^0(X)$ i.e. 0-cycles which are rationally equivalent to intersection of d many divisors of degree 0, where the equality follows from [3, Proposition 4].

Hence $[\Gamma]_* : \cap_{i=1}^d \text{Pic}^0(X) \rightarrow \cap_{i=1}^d \text{Pic}^0(X)$ is 0 map. \square

In the remaining part of this section, we apply the Theorem 2.6.1 to particular symmetrically distinguished cycles.

Corollary 2.6.3. *Let $f : X \rightarrow X$ be an automorphism of X where X is a complex abelian variety of dimension d satisfying the assumption $(*)$ as in the Section 2.3. Suppose $f_* : \frac{H_d(X)}{\widetilde{N}^1 H_d(X)} \rightarrow \frac{H_d(X)}{\widetilde{N}^1 H_d(X)}$ is the induced morphism which is Id. Then $f_* = \text{Id} : \cap_{i=1}^d \text{Pic}^0(X) \rightarrow \cap_{i=1}^d \text{Pic}^0(X)$ induced by the restriction of $f_* : CH_0(X) \rightarrow CH_0(X)$.*

Proof. Take $\Gamma = \Gamma_f - \Delta_X$ in the Theorem 2.6.1. Now Γ is symmetrically distinguished, which follows from the Remark 2.5.3(1). Hence, the corollary follows from Theorem 2.6.1. \square

Remark 2.6.4. 1. The Corollary 2.6.3 is non-trivial, only in the Cases 1, 3, 4, 5 in the List in Section 2.4, because in the Case 2, there are no non-trivial automorphisms of the abelian varieties.

2. In view of the Remark 2.5.3, it is an interesting question to write down examples of symmetrically distinguished cycles which satisfy the assumptions of the Theorem 2.6.1.

2.7 Examples

In this section, we will see some examples where the assumptions of the Corollary 2.6.3 hold true.

Example 2.7.1.

Let E be an elliptic curve over \mathbf{C} given by \mathbf{C}/Λ for a lattice Λ in \mathbf{C} . Let $X = E^d = \mathbf{C}^d/\Lambda^d$. Let $A \in SL_d(\mathbf{Z})$ be a non-trivial element of the special linear group $SL_d(\mathbf{Z})$. Now A acts naturally on \mathbf{C}^d such that $A(\Lambda^d) \subset \Lambda^d$. Thus A induces $f \in \text{Aut}(X)$. Let $\gamma_1, \dots, \gamma_d$ in \mathbf{C} be the eigenvalues of A . Given co-ordinates z_1, \dots, z_d on \mathbf{C}^d , we get dz_1, \dots, dz_d as \mathbf{C} -basis for $H_{1,0}(X) = H^0(X, \Omega_X)$. Now f induces action on the \mathbf{C} -vector space $H_{1,0}(X)$ with eigenvalues $\gamma_1, \dots, \gamma_d$. Hence, f_* acts on $H_{d,0}(X) \simeq \wedge_1^d H_{1,0}(X)$ by $\det A = \gamma_1 \cdots \gamma_d = 1$. Hence, we have $f_* = \text{Id} : H_{d,0}(X) \rightarrow H_{d,0}(X)$.

Example 2.7.2. Next, we will give examples of situations where f is an automorphism of X and $f_* : H_d(X) \rightarrow H_d(X)$ is not Identity but $f_* : \frac{H_d(X)}{\widetilde{N}^1 H_d(X)} \rightarrow \frac{H_d(X)}{\widetilde{N}^1 H_d(X)}$ is Id. Let us recall some construction of abelian varieties with real multiplication. We follow the exposition from [14, 1.13.5].

Abelian varieties with real multiplication: Let K be a totally real number field such that $\deg [K : \mathbf{Q}] = d$. Let $\alpha \mapsto \alpha_{(j)} : K \rightarrow \mathbf{R}$ be the distinct \mathbf{R} -embeddings for $1 \leq j \leq d$. Let $\tau_j \in \mathbf{C}$ be fixed such that $\text{Im}(\tau_j) > 0$ for $1 \leq j \leq d$. For the ring of integers \mathcal{O} of K , define

$$\phi : \mathcal{O} \oplus \mathcal{O} \rightarrow \mathbf{C}^d \text{ by } (\alpha, \beta) \mapsto (\alpha_{(1)}\tau_1 + \beta_{(1)}, \dots, \alpha_{(d)}\tau_d + \beta_{(d)}).$$

Define $L := \text{Im } \phi$. Note that L is a lattice in \mathbf{C}^d . Define $X := \mathbf{C}^d/L$. One can define a Riemann form on X which will imply that the complex torus X is a complex abelian variety.

Now the inclusion $K \rightarrow \text{End}^0(X)$ restricts to $\mathcal{O} \rightarrow \text{End}(X)$. So that for the units of $\mathcal{O}, \mathcal{O}^* \rightarrow \text{Aut}(X)$. By Dirichlet's unit theorem, have that \mathcal{O}^* has rank $d - 1$. So one can find $\gamma \in \mathcal{O}^*$ which is of infinite order.

The element $\gamma \in \mathcal{O}^*$ defines $f_\gamma \in \text{Aut}(X)$ given by

$$f_\gamma(z_1, \dots, z_d) = (\gamma_{(1)}z_1, \dots, \gamma_{(d)}z_d).$$

Now we will consider the action of f_γ on the cohomology of X . f_γ acts on X via a matrix $M_\gamma \in GL_d(\mathbf{C})$ with eigenvalues $\{\gamma_{(1)}, \dots, \gamma_{(d)}\}$. So the eigenvalues of f_γ on $H_d(X)$ are given by various products of these eigenvalues which in general need not be 1, which implies that f_γ is not Id on $H_d(X)$. And the eigenvalue of f_γ on $H_{d,0}(X)$ is $\gamma_{(1)}\gamma_{(2)} \cdots \gamma_{(d)} = \det M_\gamma$, which we can arrange to be 1.

More concretely, let $K = \mathbf{Q}(\sqrt{7})$ and so its ring of integers is $\mathcal{O} = \mathbf{Z}[\sqrt{7}]$. Here $d = 2 = \dim[K : \mathbf{Q}]$. Take the fundamental unit $\gamma = 8 + 3\sqrt{7}$ of \mathcal{O} . So $\gamma_{(1)} = 8 + 3\sqrt{7}$ and $\gamma_{(2)} = 8 - 3\sqrt{7}$. So the determinant of the matrix $M_\gamma = 1$ (We have taken γ such that the norm $N_{K/\mathbf{Q}}(\gamma) = 1$). But eigenvalues of M_γ on $H_2(X)$ are $\{\gamma_{(1)}^2, \gamma_{(2)}^2, 1\}$. One can check $\gamma_{(1)}^2 > 1$.

So we are in a situation where the automorphism f_γ of a complex abelian surface X acts non-trivially on $H_2(X)$, but acts trivially on $H_{2,0}(X)$.

In the next section, we discuss some applications.

3 Applications

3.1 Cycles using Pontryagin Product

For an abelian variety X of dimension d over k , an algebraically closed subfield of \mathbf{C} , S. Bloch [4, Lemma 1.2(c)] proved that $G^d := \text{Pic}^0(X)^{\cap d}$ is generated by expressions of the form

$$((a_1) - (0)) * ((a_2) - (0)) * \cdots * ((a_d) - (0))$$

where $a_1, \dots, a_d \in X(k)$ and $\gamma * \gamma'$ is the Pontryagin product of cycles γ and γ' . Thus, for f as in Corollary 2.6.3, we get identities

$$((a_1) - (0)) * ((a_2) - (0)) * \cdots * ((a_d) - (0)) = f_*(((a_1) - (0)) * ((a_2) - (0)) * \cdots * ((a_d) - (0))) \quad (4)$$

$$= ((f(a_1)) - (0)) * ((f(a_2)) - (0)) * \cdots * ((f(a_d)) - (0)). \quad (5)$$

In particular, for $d = 2$, f an automorphism of a complex abelian surface X such that f_* acts trivially on $H_{2,0}(X)$, we have identities

$$(f(a + b)) - (f(a)) - (f(b)) = (a + b) - (a) - (b) \text{ in } CH_0(X).$$

The main point of this particular case is that, a priori the two cycles on LHS and RHS are different, but still rationally equivalent to each other in $CH_0(X)$.

3.2 Action of Automorphisms of Kummer surfaces or generalized Kummer varieties on CH_0

Here we consider complex projective Kummer surfaces and their higher dimensional analogs called *generalized* Kummer varieties. Huybrechts [15]-Voisin [28] have proved the following theorem:

Theorem 3.2.1. *Let f be an automorphism of projective K3 surface X of finite order which is symplectic (i.e. which acts as Id on $H^{2,0}(X)$), then f_* acts as Id on $CH_0(X)$.*

We would like to prove a similar theorem for Kummer surface $Km(X)$ associated to a complex abelian surface X , but for (possibly infinite order) automorphisms induced from the complex abelian surface X . First let us setup some notation. Let $T(Y)$ denote the albanese kernel for a smooth, complex projective variety Y .

Kummer surface Let us recall the definition of $Km(X)$: Let $\iota : X \rightarrow X$ be the inversion map on X . So the finite group $G := \{id, \iota\}$ naturally acts on X with fixed point set given by 2-torsion points of X . Thus the quotient scheme X/G has singularity set given by the 16 2-torsion points. Now we can blow up X/G along the closed subscheme formed by these 16 points, which we call the Kummer surface $Km(X)$ associated to X . Now let f be an automorphism of abelian surface X . Thus f commutes with ι . Hence f fixes the fixed point set of G . So f lifts to an automorphism \tilde{f} of X/G which fixes the closed subscheme given by the fixed points of G . Hence one can lift the automorphism \tilde{f} to that of the Blow up $Km(X)$ given by g . So we get the commutative diagram

$$\begin{array}{ccc} Km(X) & \xrightarrow{g} & Km(X) \\ \downarrow & & \downarrow \\ X/G & \xrightarrow{\tilde{f}} & X/G \end{array} \quad (6)$$

Now we have a theorem by Bloch [6, Corollary A.10] which says that the rational map $\phi : X \dashrightarrow Km(X)$ induces a surjective isogeny $\phi^* : T(Km(X)) \rightarrow T(X)$. More precisely $\phi^* \circ \phi_* = 2Id : T(X) \rightarrow T(X)$, and also $\phi_* \circ \phi^* = 2Id$. Further Roitman's result [24] that for any smooth projective surface Y , $T(Y)$ is torsion free, implies that $\phi^* : T(Km(X)) \rightarrow T(X)$ is actually an isomorphism. Also the automorphism g commutes with the rational map ϕ . Therefore have the commuting diagram:

$$\begin{array}{ccc} T(Km(X)) & \xrightarrow{g_*} & T(Km(X)) \\ \downarrow \phi^* & & \downarrow \phi^* \\ T(X) & \xrightarrow{f_*} & T(X) \end{array} \quad (7)$$

where the vertical arrows are isomorphisms.

Now let us assume that f satisfies the assumptions of the Corollary 2.6.3, then by the conclusion $f_* : T(X) \rightarrow T(X)$ is Id. Hence by above discussion $g_* : T(Km(X)) \rightarrow T(Km(X))$ is Id.

So from above discussion, we have proved the following:

Proposition 3.2.2. *Let g be an automorphism of the Kummer surface $Km(X)$ associated to the abelian surface X , which is induced from an automorphism f of X as abelian variety. Further assume that f is symplectic automorphism of X . Then g acts as Id on the albanese kernel $T(Km(X))$ of $Km(X)$.*

Remark 3.2.3. In view of the Theorem 3.2.1, the Proposition 3.2.2 deals with automorphisms of the Kummer surface which are possibly of infinite order. One can construct Kummer surfaces with automorphisms of infinite order, for example for the Kummer surface associated with the abelian surface in the Example 2 above. We can get a similar result for generalized Kummer varieties $K_n(X)$ associated to the abelian surface X , where for $n = 1$, $K_1(X) = Km(X)$.

Generalized Kummer varieties First we recall the construction of generalized Kummer varieties K_n and the result of [20] which gives Beauville type decomposition on $CH_0(K_n)$ in terms of that of $CH_0(X)$ for a complex abelian surface X .

Let $X_0^{n+1} = \text{kernel of the sum map } \mu : X^{n+1} \rightarrow X$. The symmetric group \mathfrak{S}_{n+1} acts on X^{n+1} , hence also on X_0^{n+1} . Have the quotient map $\rho : X^{n+1} \rightarrow X^{n+1}/\mathfrak{S}_{n+1}$ with restriction to X_0^{n+1} induces its own quotient map $q : X_0^{n+1} \rightarrow K_{(n)} := X_0^{n+1}/\mathfrak{S}_{n+1}$. On the other hand μ induces the sum map $\tilde{s} : X^{n+1}/\mathfrak{S}_{n+1} \rightarrow X$. One can check that $K_{(n)} = \tilde{s}^{-1}(0)$.

Let $\Delta \subseteq X^{n+1}$ be the big diagonal. Write $D := \rho(\Delta) \subseteq X^{n+1}/\mathfrak{S}_{n+1}$. Let $\tilde{\nu} : X^{[n+1]} \rightarrow X^{n+1}/\mathfrak{S}_{n+1}$ be the Hilbert-Chow morphism, with $X^{[n+1]}$ as the Hilbert scheme of X of length $n+1$ closed subschemes of X . This morphism is a map of resolution of singularities with the exceptional divisor $E := \tilde{\nu}^{-1}(D)$. Define $K_n := (\tilde{s} \circ \tilde{\nu})^{-1}(0)$.

Now the restriction of the Hilbert-Chow morphism $\tilde{\nu}$ to K_n gives $\nu : K_n \rightarrow K_{(n)}$ which is a map of resolution of singularities and the exceptional divisor $E_0 := E \cap K_n$. Set $D_0 = D \cap K_{(n)}$ and $\Delta_0 = \Delta \cap X_0^{n+1}$. Then one has $\rho(\Delta_0) = D_0$. We have following diagram of spaces:

$$\begin{array}{ccccc}
 E_0 & \xrightarrow{\quad} & E & & \\
 \downarrow \subseteq & & \downarrow \subseteq & & \\
 & K_n & \hookrightarrow & X^{[n+1]} & \\
 D_0 & \xrightarrow{\nu} & D & & \downarrow \tilde{\nu} \\
 \downarrow \subseteq & & \downarrow \subseteq & & \\
 & K_{(n)} & \hookrightarrow & X^{n+1}/\mathfrak{S}_{n+1} & \\
 & \searrow & \swarrow & & \\
 & 0 & X & \xleftarrow{\tilde{s}} &
 \end{array} \tag{8}$$

Remark 3.2.4. K_n constructed above is called a *generalized Kummer variety* associated to the abelian surface X . For $n = 1$ one can recover the Kummer surface associated to X by above construction. K_n is an irreducible, projective, symplectic variety of dimension $2n$.

Since the multiplication morphism $m : X_0^{n+1} \rightarrow X_0^{n+1}$ commutes with the action of \mathfrak{S}_{n+1} , for each $m \in \mathbf{Z}$,

$$q^* : CH_0(K_{(n)}) \xrightarrow{\sim} \left(\bigoplus_{s=0}^{2n} CH_0(X_0^{n+1})_s \right)^{\mathfrak{S}_{n+1}} = \bigoplus_{s=0}^{2n} CH_0(X_0^{n+1})_s^{\mathfrak{S}_{n+1}}$$

Since $q_* q^* : CH_0(K_{(n)}) \rightarrow CH_0(K_{(n)})$ is given by $(n+1)!$ ([13, Lemma 1.7.6]) and the Chow groups are with rational coefficients, we get that q_* is bijective. Thus, we get

$$q_* : \bigoplus_{s=0}^{2n} CH_0(X_0^{n+1})_s^{\mathfrak{S}_{n+1}} \rightarrow CH_0(K_{(n)})$$

is bijective. We obtain a following decomposition

$$CH_0(K_{(n)}) = \bigoplus_{s=0}^{2n} CH_0(K_{(n)})_s$$

where $CH_0(K_{(n)})_s := q_* CH_0(X_0^{n+1})_s^{\mathfrak{S}_{n+1}}$. Now the Hilbert-Chow morphism $\nu : K_n \rightarrow K_{(n)}$ given by the desingularization induces an isomorphism $\nu_* : CH_0(K_n) \rightarrow CH_0(K_{(n)})$ which induces a decomposition on $CH_0(K_n)$.

$Aut(X)$ and $Aut(K_n)$: Next we want to compare automorphisms of X and K_n . Let $f \in Aut(X)$ as abelian variety. We assign to f a unique $f_n \in Aut(K_n)$ as follows:

$$\begin{array}{ccccccc} K_n & \xrightarrow{\nu} & K_{(n)} & \xleftarrow{q} & X_0^{n+1} & \longrightarrow & X^{n+1} \xrightarrow{\mu} X \\ \downarrow f_n & & \downarrow f_{(n)} & & \downarrow f_0 & & \downarrow f^{\times(n+1)} \downarrow f \\ K_n & \xrightarrow{\nu} & K_{(n)} & \xleftarrow{q} & X_0^{n+1} & \longrightarrow & X^{n+1} \xrightarrow{\mu} X \end{array} \quad (9)$$

Now as we did for the Kummer surface,

Proposition 3.2.5. *Let f_n be the automorphism of K_n induced from an automorphism f of X . Suppose f is symplectic on X . Then f_n is symplectic on K_n and f_n acts as Id on $CH_0(K_n)$.*

Proof. • f_n is symplectic: We have $H^2(K_n, \mathbf{C}) \simeq H^2(X, \mathbf{C}) \oplus \mathbf{C}[E_0]$. Further $H^{2,0}(K_n) \simeq H^{2,0}(X)$, which is functorial. Hence if f acts as Id on $H^{2,0}(X)$, then f_n acts as Id on $H^{2,0}(K_n)$. Thus, f_n is symplectic.

- By the Corollary 2.6.3, f acts as Id on $CH_0(X)_2$. Next the map $f^{\times(n+1)} : X^{n+1} \rightarrow X^{n+1}$ induces an automorphism f_0 of X_0^{n+1} . Consider the action of f_0 on $CH_0(X_0^{n+1})^{\mathfrak{S}_{n+1}}$.

We will identify X_0^{n+1} with X^n via

$$X_0^{n+1} \rightarrow X^n : (z_1, z_2, \dots, z_n, -\sum_{j=1}^n z_j) \mapsto (z_1, z_2, \dots, z_n).$$

\mathfrak{S}_{n+1} acts on X^n , via the action on X_0^{n+1} via the above isomorphism. Write \mathfrak{S}_m as acting on the set $\{1, 2, \dots, m\}$, and view $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$.

Action of \mathfrak{S}_{n+1} on X^n : \mathfrak{S}_n acts on X^n by permuting the co-ordinates. Write $t_i \in \mathfrak{S}_{n+1}$ for the transposition $(i, n+1)$ which permutes the i with $n+1$, for $1 \leq i \leq n$; the action of t_i on X^n defined by

$$t_i \cdot (z_1, z_2, \dots, z_n) \mapsto (z_1, z_2, \dots, z_{i-1}, -\sum_{j=1}^n z_j, z_{i+1}, \dots, z_n).$$

First we identify cycles in $CH_0(X^n)_{\mathfrak{S}_{n+1}}$ in terms of the cycles in $CH_0(X)_s$ for $s = 0, 2$, by the following lemma.

Lemma 3.2.6. *For $r = 0, 2, \dots, 2n$,*

$$\bigoplus_{\substack{s_1+s_2+\dots+s_n=r \\ s_j=0,2}} CH_0(X)_{s_1} \otimes CH_0(X)_{s_2} \otimes \dots \otimes CH_0(X)_{s_n} \rightarrow CH_0(X^n)_{\mathfrak{S}_{n+1}}$$

is surjective.

Proof. By Theorem 2.2.2 and 1, we have $CH_0(X^n)_r = (\pi_r^{X^n})_* CH_0(X^n)$, where

$$\pi_r^{X^n} := \sum_{s_1+\dots+s_n=r} \pi_{s_1}^X \otimes \dots \otimes \pi_{s_n}^X.$$

Note that any 0-cycle in $CH_0(X^n)$ is given by a finite sum of 0-cycles of the form $p_1^*(z_1) \cdot p_2^*(z_2) \cdots p_n^*(z_n)$, where z_i are 0-cycles in $CH_0(X)$. Let $\alpha \in CH_0(X^n)_r = (\pi_r^{X^n})_* CH_0(X^n)$, be given by $(\pi_r^{X^n})_*(p_1^*(z_1) \cdot p_2^*(z_2) \cdots p_n^*(z_n))$.

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left((\pi_r^{X^n})_*(p_1^*(z_1) \cdot p_2^*(z_2) \cdots p_n^*(z_n)) \right) = \sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left(\sum_{s_1 + \cdots + s_n = r} (\pi_{s_1}^X \otimes \cdots \otimes \pi_{s_n}^X)_*(p_1^*(z_1) \cdot p_2^*(z_2) \cdots p_n^*(z_n)) \right) \quad (10)$$

$$= \sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left(\sum_{s_1 + \cdots + s_n = r} p_1^*((\pi_{s_1}^X)_*(z_1)) \cdots p_n^*((\pi_{s_n}^X)_*(z_n)) \right) \quad (11)$$

$$= \sum_{s_1 + \cdots + s_n = r} \sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left(p_1^*((\pi_{s_1}^X)_*(z_1)) \cdots p_n^*((\pi_{s_n}^X)_*(z_n)) \right) \quad (12)$$

We claim that: if one of s_i is odd, then

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left(p_1^*((\pi_{s_1}^X)_*(z_1)) \cdots p_n^*((\pi_{s_n}^X)_*(z_n)) \right) = 0$$

Let us write $\beta_i = (\pi_{s_i}^X)_*(z_i)$ to ease the notation.

Proof of claim: First we observe that,

$$\iota^*(\beta_i) = \begin{cases} -\beta_i & \text{if } s_i = \text{odd} \\ \beta_i & \text{if } s_i = \text{even} \end{cases} \quad (13)$$

In other words, β_i is anti ι -invariant if s_i is odd and ι -invariant otherwise. Further $p_i \circ t_i = -\mu$ and $p_j \circ t_i = p_j$ for $i \neq j$. Assume that β_i is anti ι -invariant. Thus,

$$t_i^* \left(p_1^*(\beta_1) \cdots p_n^*(\beta_n) \right) = t_i^* (p_1^*(\beta_1)) \cdots t_i^* (p_n^*(\beta_n)) \quad (14)$$

$$= (-\mu)^*(\beta_i) \cdot p_1^*(\beta_1) \cdots \widehat{p_i^*(\beta_i)} \cdots p_n^*(\beta_n) \quad (15)$$

$$= -[p_1^*(\beta_i) + \cdots + p_n^*(\beta_i)] p_1^*(\beta_1) \cdots \widehat{p_i^*(\beta_i)} \cdots p_n^*(\beta_n) \quad (16)$$

The equality of (15) and (16) holds follows as β_i is anti ι -invariant [20, Lemma 4.3]. Note that

$p_j^*(\beta_i) p_1^*(\beta_1) \cdots \widehat{p_i^*(\beta_i)} \cdots p_n^*(\beta_n) = 0$ for $j \neq i$. So we have

$$t_i^* \left(p_1^*(\beta_1) \cdots p_n^*(\beta_n) \right) = -p_1^*(\beta_1) \cdots p_i^*(\beta_i) \cdots p_n^*(\beta_n).$$

Hence

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left(p_1^*(\beta_1) \cdots p_n^*(\beta_n) \right) = \sum_{\sigma \in \mathfrak{A}_{n+1}} \sigma^* \left(p_1^*(\beta_1) \cdots p_n^*(\beta_n) + t_i^* \left(p_1^*(\beta_1) \cdots p_n^*(\beta_n) \right) \right) = 0,$$

where $\mathfrak{A}_{n+1} \subset \mathfrak{S}_{n+1}$ is the alternating subgroup on $n+1$ letters. Thus the claim follows.

Therefore we concluded that

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^*(\alpha) = \sum_{\substack{s_1 + s_2 + \cdots + s_n = r \\ s_j = 0, 2}} \sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left(p_1^*((\pi_{s_1}^X)_*(z_1)) \cdots p_n^*((\pi_{s_n}^X)_*(z_n)) \right).$$

Now enough to observe that $\sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma^* \left(p_1^*((\pi_{s_1}^X)_*(z_1)) \cdots p_n^*((\pi_{s_n}^X)_*(z_n)) \right)$ is in the image of

$$\bigoplus_{\substack{s_1+s_2+\cdots+s_n=r \\ s_j=0,2}} CH_0(X)_{s_1} \otimes CH_0(X)_{s_2} \otimes \cdots \otimes CH_0(X)_{s_n} \xrightarrow{p_1^*(-) \cdots p_n^*(-)} CH_0(X^n)_r^{\mathfrak{S}_{n+1}}.$$

The lemma follows. \square

Remark 3.2.7. It was shown that $CH_0(K_n)_r \simeq q_* CH_0(X_0^{n+1})_r^{\mathfrak{S}_{n+1}} \simeq CH_0(X^n)_r^{\mathfrak{S}_{n+1}} = 0$ for r odd, by [20, Theorem 1.4].

Back to proof of Prop. 3.2.5: Since f_0 acts on $CH_0(X_0^{n+1})^{\mathfrak{S}_{n+1}}$ via the action of f on

$$\bigoplus_{\substack{s_1+s_2+\cdots+s_n=r \\ s_j=0,2}} CH_0(X)_{s_1} \otimes CH_0(X)_{s_2} \otimes \cdots \otimes CH_0(X)_{s_n},$$

which is Id as f acts as Id on $CH_0(X)_s$ for $s = 0, 2$.

Hence f_0 acts on $CH_0(X^n)_r^{\mathfrak{S}_{n+1}}$ as Id. Thus f_n acts on $CH_0(K_n)_r$ by Id for r even.

In view of the Remark 3.2.7(1) and the above conclusion, we have that f_n acts as Id on $CH_0(K_n)$. \square

Remark 3.2.8. The above proposition is in accord with a general conjecture made in [12, Conjecture 0.3], for finite order symplectic automorphisms of irreducible holomorphic symplectic varieties. Further, the automorphism f_n of K_n above could possibly be of infinite order.

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